1 Introduction

A (connected) Riemannian manifold M is *homogeneous* if M possesses a transitive action of some Lie group. When a Lie group G acts transitively on a Riemannian manifold M we can identify M with the set G/K of left cosets of the isotropy group K of a point $x_0 \in M$. (To be precise $K = \{g \in G; gx_0 = x_0\}$ and the map $G/K \to M$ is $gK \to gx_0$). The set $\{K\}$ is called the *origin* of $M = G/K$. Let $\bigtriangledown$ be an affine connection on M. We are concerned with geodesics on $(M, \bigtriangledown)$, w.r.t one parameter subgroups of G, which are called *homogeneous geodesics.* In particular, if $\bigtriangledown$ is a torsion-free affine connection on a naturally reductive homogeneous manifold $M = G/K$, then all geodesics on M are homogeneous [12],(or see [8], vol.II, pp.196-197). If g is a G-invariant Riemannian metric on $M = G/K$ then with the Levi-Civita connection $\bigtriangledown$ arising from g, all geodesics on M are homogeneous [11]. Every semi-Riemannian group space G admits at least one homogeneous geodesic passing through the identity element $e \in G$, [6]. It is proved in [9], that every Riemannian homogeneous manifold M at any point $x_0 \in M$ has at least one homogeneous geodesic.

2 Homogeneous and associated bundles

A smooth map π between differentiable manifolds M and N will be said to have the *local product property* with respect to a differentiable manifold F if there is an open covering U_α of N and a family $\{\psi_\alpha\}$ of diffeomorphisms

$$\psi_\alpha : U_\alpha \times F \longrightarrow \pi^{-1}(U_\alpha)$$

such that $\pi\psi_\alpha(x, y) = x$, where $x \in U_\alpha$, $y \in F$.
The system $\{(U_\alpha, \psi_\alpha)\}$ will be called *local decomposition* of π.

A four-tuple (M, π, N, F) is called a *smooth fiber bundle* if π be a smooth map with local product property with respect to a differentiable manifold F. A local product property for π is called *coordinate representation* for the fiber bundle.
We call M the *bundle space* , N the *base space* and F the *fiber space* of a smooth fiber bundle. A *cross section* of (M, π, N, F) is a smooth map $\sigma : N \to M$ s.t $\pi \circ \sigma = id_N$.

Definition 2.1. A (smooth)*principal bundle with structure group G* is a pair $(\wp, T)$,where
(i) $\wp = (P, \pi, B, G)$ is a smooth fiber bundle.
(ii) $T : P \times G \longmapsto P$ is a right action of G on P.

(iii) $\wp$ admits a coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ such that

$$\psi_\alpha(x, ab) = \psi_\alpha(x, a)b, \quad x \in U_\alpha, \quad a, b \in G.$$

The action T is called the *principal action* and coordinate representation $\{(U_\alpha, \psi_\alpha)\}$ is called a *principal coordinate.*

Let $T : G \times M \longrightarrow M$ be a transitive action of the Lie group G on a differentiable manifold M, and let K be the invariant subgroup of the point $x_0 \in M$. Then by the map $\phi : G/K \longrightarrow M$ with $\phi(gK) = gx_0$, one can take $M = G/K$ as a homogeneous differentiable manifold.
Consider the canonical map $\pi : G \to G/K$ given by $\pi(g) = gK$, it is obvious that π is a smooth map between differentiable manifolds G and G/K with local product property with respect to a differentiable manifold K. Then $\Im = (G, \pi, G/K, K)$ is a smooth fiber bundle and together with the action of K on G by right multiplication, is a principal bundle with structure group K (see definition 2.1).
Now we give the following

Definition 2.2. Let K be a closed subgroup G, the principal fiber bundle $\Im = (G, \pi, G/K, K)$, is called *principal homogeneous bundle.*

Definition 2.3. Let $\wp = (P, \pi, B, G)$ be a principal bundle and F be a differentiable manifold, consider the left action Q, of G on the product manifold $P \times F$ given by

$$Q_a(z, y) = (z, y).a = (z.a, a^{-1}.y) \quad z \in P, y \in F, a \in G.$$

Q is called the *joint action of* G.

The set of the orbits for the joint action is denoted by $P \times_G F$ and

$$q : P \times F \to P \times_G F$$

will denote the corresponding projection, i.e. $q(z, y)$ is the orbit through (z, y).
The map q determines a map $\rho : P \times_G F \to B$ such that,

$$\rho \circ q = \pi \circ \pi_p.$$

Where, $\pi_p : P \times F \to P$ is the canonical projection and $\pi : P \to B$ is the bundle map.

We consider a smooth structure on $P \times_G F$. Let $\{U_\alpha\}$ be an open covering of B and $\sigma_\alpha : U_\alpha \to P$ be a cross section.
These are related by

$$\sigma_\beta(x) = \sigma_\alpha(x).g_{\alpha\beta}(x).$$

Where $x \in U_\alpha \cap U_\beta$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are smooth maps. We define set maps,

$$\varphi_\alpha : U_\alpha \times F \to \rho^{-1}(U_\alpha),$$

by setting

$$\varphi_\alpha(x, y) = q(\sigma_\alpha(x), y), \quad x \in U_\alpha, y \in F.$$

Then $\rho(\varphi_\alpha(x, y)) = x$ and so φ_α is restricted to set maps,

$$\varphi_{\alpha\beta} : F \to \rho^{-1}(x) \quad x \in U_\alpha.$$

Moreover, to each orbit in $\rho^{-1}(x)$ there corresponds a unique $y \in F$ such that the orbit passes through $(\sigma_\alpha(x), y)$.
Hence $\varphi_{\alpha x}$ is bijective, and so φ_α is bijective.
Further, the relation $q(z.a, y) = q(z, a.y)$ implies that

$$\varphi_\alpha^{-1} \circ \varphi_\beta(X, Y) = (x, g_{\alpha\beta}(x).y), \quad x \in U_\alpha \cap U_\beta, y \in F.$$

Thus by, [4] sec. 1.13, proposition X, ξ is a smooth principal bundle with coordinate representation $\{(U_\alpha, \psi_\alpha)\}$.
In this way, we prove the following

Proposition 2.4. *There is a unique smooth structure on $P \times_G F$, such that $\xi = (P \times_G F, \rho, B, F)$ is a smooth fiber bundle.*

Definition 2.5. $\xi = (P \times_G F, \rho, B, F)$, is called the *fiber bundle associated with $\wp = (P, \pi, B, G)$.*

Lemma 2.6. *Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle then*

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathrm{G/K}, \mathcal{G}/\mathcal{K}),$$

will be an associated bundle of $\Im = (G, \pi, G/K, K)$.

Proof. Recall that G is a connected Lie group and K is a closed subgroup of G. Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$. Let $\Im = (G, \pi, G/K, K)$, is a principal homogeneous bundle, the following diagram is commutative

$$G \times \mathcal{G}/\mathcal{K} \to^q G \times_\mathcal{K} \mathcal{G}/\mathcal{K}$$

$$\downarrow_{\pi_G} \qquad\qquad \downarrow_{\rho_\xi}$$

$$G \quad \rightarrow_\pi \quad G/K.$$

Proposition 2.4, implies that there is a unique smooth structure on $G\times_K \mathcal{G}/\mathcal{K}$ such that $\xi = (G\times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$, be a smooth fiber bundle. By definition 2.5,

$$\xi = (G\times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}),$$

is an associated bundle of $\Im = (G, \pi, G/K, K)$.$\diamond$

3 Group structure

In this section we give some results about Lie group G and its Lie algebra $\mathcal{G}$. Recall that the *commutator group* of a group G is the subgroup $[G, G] = G'$ generated by all the commutators $[x, y] = xyx^{-1}y^{-1}$, where $x, y \in G$.
The *iterated commutator groups* G^i (i=1, 2, 3,...) of G are defined by induction

$$G^0 = G,\ \ G' = [G, G],\ \ G'' = [G', G'],\ \ G^{i+1} = [G^i, G^i]\ \text{ etc.}$$

A group G is called *solvable group* if there exists j such that
$G^j = \{e\}$.
For any group G, the *lower central series* G^i (i=1, 2, 3,...) of G is defined by induction

$$G_0 = G,\ \ G_1 = [G, G],\ \ G_2 = [G_1, G],\ \ G_{i+1} = [G_i, G]\ \text{ etc.}$$

A group G is called *nilpotent group* if there exists j such that
$G_j = \{e\}$.
Clearly, $G^i \subset G_i$, hence every nilpotent group is solvable.

Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Consider the *commutator series* of $\mathcal{G}$, defined recursively by

$$\mathcal{G}^0 = \mathcal{G},\ \ \mathcal{G}' = [\mathcal{G}, \mathcal{G}],\ \ \mathcal{G}'' = [\mathcal{G}', \mathcal{G}'],\ \ \mathcal{G}^{i+1} = [\mathcal{G}^i, \mathcal{G}^i]\ \text{ etc.}$$

$\mathcal{G}$ is called *solvable* if there exists j such that $\mathcal{G}^j = \{o\}$.

Theorem 3.1. *Every Lie algebra $\mathcal{G}$ has the largest solvable ideal and every Lie group G has the largest connected normal solvable Lie subgroup such that its Lie algebra coincides with the largest solvable ideal of the algebra $\mathcal{G}$* (see [3], page 55).

The subgroup of a Lie group G in theorem 3.1 is called the *radical of* G and is denoted by $RadG$.
The largest solvable ideal of the algebra $\mathcal{G}$ is called the *radical of* $\mathcal{G}$ and is denoted by $rad\mathcal{G}$.

Definition 3.2. Let C be a bilinear symmetric form on a finite dimensional vector space V. The *radical* of C is a vector subspace of V s.t

$$radC = \{v \in V; C(v,u) = 0 \qquad \forall u \in V\}.$$

Let $\mathcal{G}$ be a Lie algebra *adjoint representation* of $\mathcal{G}$ is given by

$$ad_X(Y) = [X,Y], \quad X, Y \in \mathcal{G}.$$

Definition 3.3. Recall that the *Killing form* B on $\mathcal{G}$ is the bilinear symmetric form defined by $B(X,Y) = Tr(ad_X ad_Y)$ (see [7], page 669).

Definition 3.4. Lie group G is called *semisimple* if $RadG = \{e\}$ and the Lie algebra $\mathcal{G}$ is called *semisimple* if $rad\mathcal{G}$ $=\{0\}$.

If a Lie algebra $\mathcal{G}$ is semisimple, then $[\mathcal{G},\mathcal{G}] = \mathcal{G}$ and it is well known, that $\mathcal{G}$ is semisimple iff B is non degenerate (see [7], page 669).

Theorem 3.5. *Every Lie algebra $\mathcal{G}$ has the largest nilpotent ideal, moreover every Lie group G has the largest connected normal and nilpotent subgroup such that its Lie algebra is the largest nilpotent ideal in $\mathcal{G}$* (see [3], page 59).

Definition 3.6. The largest nilpotent ideal in a Lie algebra $\mathcal{G}$ is called the *weak radical of* $\mathcal{G}$ and is denoted by $W_{rad}\mathcal{G}$. The largest Lie subgroup which is normal nilpotent and its Lie algebra is $W_{rad}\mathcal{G}$ is called *weak radical of* G and is denoted by $W_{Rad}G$.

By theorems 3.1 and 3.5 we have the following

$$W_{rad}\mathcal{G} \subseteq rad\ \mathcal{G}$$
$$W_{Rad}G \subseteq RadG.$$

Definition 3.7. If B is the Killing form on $\mathcal{G}$, then we define the *weak radical of* B as follows

$$W_{rad}B = \{X \in [\mathcal{G},\mathcal{G}], B(X,Y) = 0 \quad \forall Y \in \mathcal{G}\}.$$

Definition 3.8. A Lie group G (Lie algebra $\mathcal{G}$) is called *weakly semisimple* if $W_{Rad}G = \{e\}$,($W_{rad}\mathcal{G} = \{0\}$).

4 Homogeneous geodesics and homogeneous vectors

Let M be a differentiable manifold and $\chi(M)$ be a set of all differentiable vector fields on M. An *affine connection* on M is a rule ∇ which assigns to each $X \in \chi(M)$ a linear mapping $\nabla_X : \chi(M) \to \chi(M)$ satisfying the following conditions

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$$

$$\nabla_X(fY) = f\nabla_X(Y) + (Xf)Y$$

where $f, g \in C^1(M)$ and $X, Y \in \chi(M)$ (see [2], chap.2). The operator ∇_X is called *covariant differentiation* with respect to X. The differentiable curve $\gamma : t \to \gamma(t)$ in M is called a *geodesic* if $\nabla_{d\gamma/dt}(d\gamma/dt) = o$ (see [2], sec.3, chap.2).
Let G be a Lie group and $\varphi_t : M \to M$, $t \in G$ be a diffeomorphism induced by action $T : G \times M \longrightarrow M$. Then the affine connection ∇ on M is invariant under action T if $d\varphi_t(\nabla_X Y) = \nabla_{d\varphi_t(X)} d\varphi_t(Y)$.
Let $X \in \mathcal{G}$, the trajectories of X determine a mapping $\varphi : R \to G$ with $\varphi(0) = e$ and $\varphi'(t) = X_{\varphi(t)}$, φ is called *1-parameter subgroup* of G.
Where T_xM be a tangent space of differentiable manifold M. As we know in [2], chap.3, given a point x of M and a vector $v \in T_xM$ there exists unique parameterized geodesic $\gamma : I \in \mathbf{R} \to M$, with $\gamma(0) = x$ and $\gamma'(0) = v$. To indicate the dependence of this geodesic on the vector v, it is convenient to denote it by $\gamma(t, v) = \gamma(t)$.
If $v \in T_xM$, we define exponential map as fallows

$$exp : T_xM \to M$$

$$exp_x(v) = \gamma(1, v), \quad exp_x(0) = p.$$

Recall that G is a connected Lie group and K is a closed subgroup of G. The set of left cosets of K in G is denoted by G/K and can be given a unique differentiable structure, then $M = G/K$ is called a homogeneous manifold.
We shall now introduce the following

Definition 4.1. Let $\bigtriangledown$ be an Affine connection on $M = G/K$. Let $\bigtriangledown$ be invariant under the natural action of $T : G \times M \longrightarrow M$. Then a geodesic $\gamma : I \longrightarrow M$ is called *homogeneous* if, there exists a 1-parameter subgroup $t \longrightarrow \exp tX$, $t \in R$, of G with $X \in \mathcal{G}= T_eG$
such that

$$\gamma(t) = T(\exp tX, x).$$

Where $\gamma(0) = x \in M$, $\quad t \in I \subset \mathbf{R}$ and $exp : \mathcal{G} \to G$ is the exponential map.

Definition 4.2. Let G be a connected Lie group and K a closed subgroup of G and $\mathcal{G}$ be a Lie algebra of G. The vector $0 \neq X \in \mathcal{G}$ is called a *homogeneous vector* (or *geodesic vector*), if the curve $\gamma(t) = (\exp tX)(x_0)$ is a geodesic on $M = G/K$ (see [11]).

Definition 4.3. Denote by $\mathcal{G}$ and $\mathcal{K}$ the Lie algebras of G and K respectively and consider the adjoint representation $Ad : K \times \mathcal{G} \to \mathcal{G}$. Let $M = G/K$ be a homogeneous space. The homogeneous space G/K is *reductive* if the Lie algebra $\mathcal{G}$ is the direct some of the Lie algebra $\mathcal{K}$ and a vector subspace $\mathcal{M}$ of $\mathcal{G}$ which is $Ad(K)$-invariant, i.e.

1. $\mathcal{G} = \mathcal{M} + \mathcal{K} \qquad \mathcal{M} \bigcap \mathcal{K} = \{0\}$
2. $Ad(K)(\mathcal{M}) \subset \mathcal{M}$ (see [8], vol.II, p.190).

The following result can be found in [11], proposition 1.

Any homogeneous Riemannian manifold G/K has the reductive decomposition of the form

$$\mathcal{G} = \mathcal{M} + \mathcal{K}$$

where $\mathcal{M} \subset \mathcal{G}$ is a vector subspace such that $Ad(K)(\mathcal{M}) \subset \mathcal{M}$.

Now let $M = G/K$ be a reductive homogeneous Riemannian manifold and $\mathcal{G} = \mathcal{M} + \mathcal{K}$, its Lie algebra decomposition. The natural map $\phi : G \longrightarrow \frac{G}{K} = M$ will induce a linear epimorphism $(d\phi)_e : T_eG \longrightarrow T_{x_0}M$, and the vector space $\mathcal{M}$ will be identified with $T_{x_0}M$.

Definition 4.2 implies a correspondence between the homogeneous vectors, and homogeneous geodesics passing through $x_0 \in M$. Now if C is a scalar product on $\mathcal{M}$ induced by the scalar product on $T_{x_0}M$, then the following lemma holds

Lemma 4.4. *If X belong to $\mathcal{G}$, let $[X,Y]_{\mathcal{M}}$ and $X_{\mathcal{M}}$ be the components of $[X,Y]$ and X in $\mathcal{M}$ with respect to reductive decomposition. Then X is homogeneous (or geodesic) vector iff*

$$C(X_{\mathcal{M}}, [X,Y]_{\mathcal{M}}) = 0 \quad \forall Y \in \mathcal{G}, (\textit{see } [11], \ \textit{proposition } 2.1).$$

A finite family $\{\gamma_1, \gamma_2, \dots \gamma_n\}$ of homogeneous geodesics through $x_o \in M$ is said to be *orthogonal* (or *linearly independent*, respectively) if the corresponding initial tangent vectors at x_0 are orthogonal (or *linearly independent*, respectively) the following result is obvious

Proposition 4.5. *A finite family $\{\gamma_1, \gamma_2, \dots \gamma_n\}$ of homogeneous geodesics through $x_o \in M$ is orthogonal (respectively, linearly independent) if the $\mathcal{M}$-component of the corresponding homogeneous geodesic vectors are orthogonal (respectively, linearly independent)* (see [10]).

By [9] proposition 3, part A, we can conclude the following

Proposition 4.6. *Let G be a connected Lie group and $T : G \times M \longrightarrow M$ be a transitive action of G on a Riemannian manifold M, and let K be the isotropy subgroup of a point $x_0 \in M$. If G is a solvable Lie group then there exist at least one homogeneous vector in $\mathcal{M}$ passing through x_0.*

5 Main results

Recall that G is a connected Lie group and $T : G \times M \longrightarrow M$ is a transitive action of G on a differentiable manifold M, and K is the invariant subgroup of the point $x_0 \in M$. Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$.

Theorem 5.1. *Let G be a connected transitive Lie subgroup of the isometry group $I(M)$ of a Riemannian manifold M and let $M = G/K$. Where*

$$\Im = (G, \pi, G/K, K)$$

be a principal homogeneous bundle and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathrm{G/K}, \mathcal{G}/\mathcal{K}),$$

be a bundle associated with $\Im$.Let $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$. and $x_0 = \{K\}$ be the origin of M, where $\mathcal{G}' = \mathcal{S} + \mathcal{P}$ is a reductive decomposition of $\mathcal{G}'$, If G is weakly semisimple, then r orthogonal homogeneous geodesics are passing through x_0, where $r = \dim S$,

Theorem 5.2. *By hypotheses of theorem 5.1, let*

$$\Im = (G, \pi, G/K, K)$$

be a principal homogeneous bundle and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, \mathrm{G/K}, \mathcal{G}/\mathcal{K})$$

be a bundle associated with $\Im$*. Where* V *is a vector subspace of* $\mathcal{G}$ *such that* V *is* $Ad(K)$*-invariant and irreducible with respect to the restricted adjoint representation* $Ad(K) : K \times \mathcal{G} \longrightarrow \mathcal{G}$*. And let the killing form* B *on* V *is non degenerate and* $V, (\mathcal{K})$ *are orthogonal with respect to* B*.If* G *is a semisimple Lie group, then all the members of* $V/\mathcal{K}$ *(set of left cosets* $\mathcal{K}$ *in* V*) are geodesic vectors.*

Let M be a differential manifold and $T_M = \bigcup_{a \in M} T_a M$, where $T_a M$ is the tangent space of M at a. Let

$$\pi_M : T_M \to M$$

be the canonical projection such that for all $v \in T_a M$ we have $\pi_M(v) = a$. In this case, there is a unique differentiable structure on T_M such that $\tau_M = (T_M, \pi_M, M, \mathbf{R}^m)$ is a vector bundle ([4], vol.I, chap.3), τ_M is called the *tangent bundle of M.*

Lemma 5.3. *Let* $T : G \times M \longrightarrow M$ *be a transitive action of* G *on a Riemannian manifold* M*, and let* K *be the isotropy subgroup of some point* $x_0 \in M$*. If* $\mathcal{G} = \mathcal{M} + \mathcal{K}$ *is the reductive decomposition of the Lie algebra of* G*, and* B *is the Killing form of* $\mathcal{G}$*, then* $W_{rad}B \subseteq \mathcal{M}$*.*

Proof. Let B be the Killing form on $\mathcal{G}$. We have $W_{rad}B \subseteq radB$. But B is negative definite on $\mathcal{K}$, hence $radB \subseteq \mathcal{M}$ ([9], Proposition 2). Thus $W_{rad}B \subseteq \mathcal{M}$.$\diamond$

Definition 5.4. Suppose that Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$. Then

$$\mathcal{G} = \mathcal{K}^\perp \oplus \mathcal{K}$$

where $\mathcal{K}^\perp$ is the *orthogonal complement* of $\mathcal{K}$ in $\mathcal{G}$ with respect to a bilinear form in $\mathcal{G}$ ([7], p.669).

Let $Ad : a \mapsto Ada$ be the adjoint representation of G. Then by definition 5.4 we can write

$$Ady = Ad^\perp y \oplus id_{\mathcal{K}}, \quad y \in G,$$

where $Ad^{\perp}y$ denotes the restriction of Ady to $\mathcal{K}^{\perp}$.

Theorem 5.5. *Let*

$$\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \boldsymbol{R}^m)$$

be the tangent bundle of homogeneous Riemannian manifold G/K *and* $\Im = (G, \pi, G/K, K)$*, be the homogeneous bundle. Let*

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}),$$

be the associated bundle of $\Im = (G, \pi, G/K, K)$*, and* $\mathcal{S}$ *be the vector subspace of* $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ *, in the reductive decomposition of* $\mathcal{G}'$*. If* G *is a weakly semisimple Lie group then there are* $n = dim\mathcal{S}$ *mutually orthogonal homogeneous vectors in* $\boldsymbol{R}^m$ *and if* G *is a semisimple Lie group then there are* m *mutually orthogonal homogeneous vectors in* $\boldsymbol{R}^m$*, such that* $m \geq n$ *and equality holds iff* G *is a semisimple Lie group .*

Proof. Let G be a connected Lie group and $T : G \times M \longrightarrow M$ be a transitive action of G on a Riemannian manifold M, and K be the isotropy subgroup of a point $x_0 \in M$. Lie algebras of K and G are denoted by $\mathcal{K}$ and $\mathcal{G}$.
The adjoint representation of G is restricted to a representation $Ad_{G,K}$, of K in $\mathcal{G}$. Since the Lie algebra $\mathcal{K}$ is stable under the map $Ad_{G,K}(a)$, $a \in K$, we obtain a representation $Ad^{\perp}$ of K in $\mathcal{G}/\mathcal{K}$. The projection $a \mapsto aK$ defines a surjective map $\pi : G \to G/K$, observe that $\pi(K) = e \in G$. Since $\mathcal{K} = T_e(K)$, it follows that $\mathcal{K}$ is a subset of $ker(d\pi)_e$. On the other hand

$$dim\ Im(d\pi)_e = dimT_e(G/K) = dim\mathcal{G} - dim\ \mathcal{K}.$$

Hence $ker(d\pi)_e = \mathcal{K}$. So $(d\pi)_e$ induces a linear isomorphism,

$$\mathcal{G}/\mathcal{K} \to T_e(G/K).$$

For each $a \in G$, let $L_a : G \to G$ given by $L_a(x) = ax$ be a left translation in G. We denote the differential of L_a by $dL_a : \mathcal{G} \to \mathcal{G}$. Now we take $R_a : G \to G$ as a right translation for each $a \in G$, so for each $a \in G$, we have $\pi o R_a = \pi$. On the other hand $\pi o L_a = T_a o \pi$, thus

$$(d\pi)_e o Ad_{G,K}(a) = dT_a o (d\pi)_e.$$

Thus the isomorphism

$$(d\pi)_e : \mathcal{G}/\mathcal{K} \to T_e(G/K)$$

is equivariant with respect to $Ad^{\perp}$ and dT, so by [4], vol.I, p. 45, we have a strong isomorphism between $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbf{R}^m)$ and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}).$$

Since G is connected, G' is a normal Lie subgroup of G and $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ is its Lie algebra. In the reductive decomposition with respect to the restriction of the Killing form B of $\mathcal{G}$, on $\mathcal{G}'$, we set $\mathcal{G}' = \mathcal{S} + \mathcal{P}$, where S is a subspace of $\mathcal{G}'$ with dimension n and $\mathcal{P}$ is the Lie algebra of the closed subgroup P of G' such that $P = G' \cap K$. Let B' be the restriction of Killing form B on $\mathcal{G}'$. By lemma 5.1 we have $W_{rad}B' \subseteq \mathcal{S}$. Let $W_{rad}B' = \mathcal{S}$. In this case, since $W_{rad}B' \subseteq radB'$, we have, $W_{rad}B'$ is a solvable subalgebra of $\mathcal{G}$, so there is a solvable Lie group with a transitive action on M, $\mathcal{S}$ admits at least one homogeneous vector through $x_0 \in M$.

Now we consider the case $W_{rad}B' \subset \mathcal{S}$.

Let C be a scalar product on $\mathcal{M}$ induced by the scalar product on $T_{x_0}M$, and C' be the restriction of C on $\mathcal{S}$. Then we define an endomorphisms $\phi : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$B'(X, Y) = C'(\phi(X), Y) \qquad X, Y \in \mathcal{S}.$$

(see [7], p.669).

Hence, with respect to an orthogonal basis of B', the corresponding matrices of ϕ and B', are the same. Hence the corresponding matrix of ϕ is symmetric. It means that the eigenvalues $\lambda_1, \cdots, \lambda_n$ of ϕ are all reals. Then the corresponding eigenvectors $v_1, \cdots, v_n$ form an orthogonal basis of $\mathcal{S}$ with respect to B' such that for $i \neq j$, $B'(v_i, v_j) = 0$. If for some index k, $B'(v_k, v_k) = 0$, then $v_k \in W_{rad}B'$ and so by lemma 4.2, v_k is a homogeneous vector in $\mathcal{S}$. Choose $v_k \in (\mathcal{S} - W_{rad}B')$, then λ_k is nonzero, and for any $Z \in \mathcal{G}'$ we have

$$C'(v_k, [v_k, Z]_{\mathcal{S}}) = \tfrac{1}{\lambda_k} C'(\phi(v_k), [v_k, Z]_{\mathcal{S}})$$

$$= \tfrac{1}{\lambda_k} B'(v_k, [v_k, Z]_{\mathcal{S}}) = \tfrac{1}{\lambda_k} B'(v_k, [v_k, Z])$$

$$= -\tfrac{1}{\lambda_k} B'([v_k, v_k], Z) = 0$$

Thus v_k is a homogeneous vector in $\mathcal{S}$ (see lemma 4.2).

If G is a weakly semisimple Lie group then $W_{rad}B' = 0$, since $W_{rad}B' = ker\phi$, we have $ker\phi = 0$, thus ϕ is an isomorphism of $\mathcal{S}$ and all eigenvalues $\lambda_1, \cdots, \lambda_n$ are non zero. Hence eigenvectors $v_1, \cdots, v_n$ are homogeneous vectors. By strong isomorphism between,

$$\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbf{R}^m)$$

and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}),$$

there exist n vectors $w_1, \cdots, w_n$ in fiber space of $\tau_{G/K}$ such that they are images of $v_1, \cdots, v_n$ under the isomorphism. Hence, if G is a weakly semisimple Lie group then there are n mutually orthogonal homogeneous vectors in $\mathbf{R}^m$, where $n = dim\mathcal{S}$.

Now, we complete the proof of the theorem in the case G is a semisimple Lie group. Let $\mathcal{G} = \mathcal{M} + \mathcal{K}$ be the reductive decomposition of $\mathcal{G}$. By lemma 5.1, $radB \subseteq \mathcal{M}$. Now by similar argument, we can consider two cases, $radB = \mathcal{M}$ and $radB \subset \mathcal{M}$. If $radB = \mathcal{M}$, since $radB$ is a solvable subalgebra of $\mathcal{G}$, then there exist a solvable Lie group acts transitively on M, so $\mathcal{M}$ has at least one homogeneous vector through $x_0 \in M$ (Proposition 4.4.).

In the second case by the inner product C on $\mathcal{M}$, we define an endomorphisms $\phi : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$B(X, Y) = C(\phi(X), Y) \qquad X, Y \in \mathcal{M}.$$

Then we can find eigenvalues $\lambda_1, \cdots, \lambda_m$ and the corresponding eigenvectors $v_1, \cdots, v_m$ to form an orthogonal basis of $\mathcal{M}$ with respect to B. By the first part of the proof, it is easy to show that $v_k \in (\mathcal{M} - radB)$ is a homogeneous vector in $\mathcal{G}/\mathcal{K}$. If $\mathcal{G}$ is semisimple, then $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$ and $radB = 0$. But $B(X, Y) = C(\phi(X), Y)$, hence $Ker\phi = radB$ and ϕ is isomorphism. In this case all eigenvalues $\lambda_i, \quad 1 \leq i \leq m$ are non zero and so eigenvectors $v_1, \cdots, v_m$ are all homogeneous vectors. Then by strong isomorphism we can define homogeneous vectors $w_1, \cdots, w_m$ in the fiber space of $\tau_{G/K}$.

We have $W_{rad}B \subseteq radB$, and so, if G be a semisimple Lie group, we conclude that $W_{rad}B = 0$. Hence it is obvious that $m = n$, and the proof is complete .$\diamond$

By theorem 5.5 and proposition 4.5, we have relations between homogeneous geodesics in the base space and homogeneous vectors in the fiber space of $\tau_{G/K}$ as follow

Corollary 5.6. *Let*

$$\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \boldsymbol{R}^m)$$

be the tangent bundle of homogeneous Riemannian manifold G/K. If G is a semisimple Lie group then there is a finite family $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ of orthogonal homogeneous geodesics through the origin of the base space of $\tau_{G/K}$. If G is a weekly semisimple Lie group then there are n orthogonal homogeneous geodesics through the origin of the base space of $\tau_{G/K}$, where $m \geq n$ and equality holds iff G is a semisimple Lie group.

In the following theorem we give a subspace of $\mathcal{G}'$ such that all member of this subspace are homogeneous vectors, and by strong isomorphism between $\tau_{G/K}$ and ξ we can find a subspace of $\mathbf{R}^m$ (under isomorphism) such that all members of this subspace are homogeneous vectors.

Theorem 5.7. *Let $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \boldsymbol{R}^m)$ be a tangent bundle of G/K. Let V be a vector subspace of $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$, such that V is $Ad(K)$-invariant and irreducible with respect to the restricted adjoint representation $Ad : K \times \mathcal{G} \longrightarrow \mathcal{G}$. If the Killing form B on V is non- degenerate and, V and $\mathcal{K}$ are orthogonal with respect to B, then up to isomorphism there is a subspace W of R^m such that for every w in W, w is a homogeneous vector.*

Proof. Let $\Im = (G, \pi, G/K, K)$ be the homogeneous bundle, with associated bundle $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$. We consider the subspace $\mathcal{G}'/\mathcal{P}$ of fiber space of ξ, where G' is the derived subgroup of G and $\mathcal{G}' \subset \mathcal{G}$ is its Lie algebras and let $\mathcal{G}' = \mathcal{S} + \mathcal{P}$ be the reductive decomposition of $\mathcal{G}'$. By hypotheses of theorem V and $\mathcal{K}$ are orthogonal with respect to B, hence V is a subspace of $\mathcal{S}$. Let C be the scaler product on $\mathcal{M}$ and C' be the restriction of C on $\mathcal{S}$ and B' be the restriction of Killing B on $\mathcal{P}$. We define $\psi : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$B'(X, Y) = C'(\psi(X), Y), \qquad X, Y \in \mathcal{S},$$

and for same $\lambda \in \mathbf{R}$, we get $B' = \lambda C'$ (see [8], Appendix 5). But B is non-degenerate on

$$V \subseteq \mathcal{S} \subseteq \mathcal{M}$$

(Lemma 5.1), hence $\lambda \neq 0$.

For $X \in V$, $\psi(X) = \lambda X$ hence for every $Z \in \mathcal{G}'$ as in theorem 5.3 we obtain

$$C'(X, [X, Z]_{\mathcal{S}}) = \tfrac{1}{\lambda} C'(\psi(X), [X, Z]_{\mathcal{S}})$$

$$= \tfrac{1}{\lambda} B'(X, [X, Z]_{\mathcal{S}}) = \tfrac{1}{\lambda} B'(X, [X, Z])$$

$$= -\tfrac{1}{\lambda} B'([X, X], Z) = 0.$$

Thus X is a homogeneous vector. Hence all members of $V/\mathcal{P}$ are homogeneous vectors. Now under isomorphism there is a subspace W of $\mathbf{R}^m$ such that for every w in W, w is a homogeneous vector.$\diamond$

6 EXAMPLE

Homogeneous geodesics on the solvable homogeneous principal bundles

Recall that the *iterated commutator groups* G^i (i=1, 2, 3,...) of G are defined by induction

$$G^0 = G, \ \ G' = [G, G], \ \ G'' = [G', G'], \ \ G^{i+1} = [G^i, G^i] \ \text{ etc.}$$

A group G is called *solvable group* if there exists j such that
$G^j = \{e\}$.

Theorem 6.1. *Let $\Im = (G, \pi, G/K, K)$, be a principal homogeneous bundle and $\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$, be the associated bundle of $\Im = (G, \pi, G/K, K)$. If G is the matrix group of all matrices of the form*

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix}$$

then a vector V in the fiber space of ξ is a homogeneous if and only if its components (x, y, z) satisfy

$$xz = 0, \ \ xy = 0, \ \ x^2 = y^2.$$

Proof. Let $\Im = (G, \pi, G/K, K)$ be a principal homogeneous bundle, with the associated bundle $\xi = (G \times_K \mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$. Let
G be the matrix group of all matrices of the form

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $(x, y, z) \in R^3$, the Lie group G is unimodular and solvable (see [7], pp.134-136), with the left invariant Riemannian metric

$$g = e^{2z} dx^2 + e^{-2z} dy^2 + \lambda^2 dz^2.$$

Where $\lambda > 0$ is a constant. Then G is a homogeneous Riemannian manifold with the origan at $(0, 0, 0)$ ([7], p.134).

if $\mathcal{G} = \mathcal{M} + \mathcal{K}$ be the reductive decomposition of $\mathcal{G}$ then $\mathcal{K}= 0$, and hence $\mathcal{G} = \mathcal{M}$. Let $\xi = (G \times_K \mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K})$, be the associated bundle of $\Im$, by definition 3.1, we can take $\xi = (G \times \mathcal{M}, \rho_\xi, G, \mathcal{M})$, is the associated bundle of $\Im$.

Consider the vector fields, $X = e^{-z}\partial/\partial x$, $Y = e^z\partial/\partial y$ and $Z = \partial/\partial z$. Then, $\{X, Y, Z\}$ is a basis of the Lie algebra $\mathcal{G} = \mathcal{M}$ of G.
Using such basis in the fiber space of

$$\xi = (G \times \mathcal{M}, \rho_\xi, G, \mathcal{M}),$$

we now compute explicitly the Lie bracket $[,]$ and the scaler product C on $\mathcal{G} = \mathcal{M}$. We have

$$\begin{aligned}
[X, Y] &= [e^{-z}\partial/\partial x, e^z\partial/\partial y] = 0 \\
[X, Z] &= [e^{-z}\partial/\partial x, \partial/\partial z] = e^{-z}\partial/\partial x = X \\
[Y, Z] &= [e^z\partial/\partial y, \partial/\partial z] = -e^z\partial/\partial y = -Y \\
[Z, Z] &= [\partial/\partial z, \partial/\partial z] = 0
\end{aligned}$$

and

$$\begin{aligned}
&C(X, Y) = 0 \quad C(X, X) = 1 \quad C(Y, Y) = 1 \\
&C(Z, Z) = \lambda^2 \quad C(X, Z) = 0 \quad C(Y, Z) = 0
\end{aligned}$$

If V is an element of $\mathcal{G}$, then $V = xX + yY + zZ$. Where (x, y, z) components of V. Hence on the fiber space of ξ we compute the following

$$C([V, X], V) = C(-zX, V) = -xz$$

$$C([V, Y], V) = C(zY, V) = yz$$

$$C([V, Z], V) = C(xX - yY, V) = x^2 - y^2.$$

Therefore the vector V in the fiber space of the associated bundle of $\Im$, is homogeneous if and only if, its component (x, y, z) satisfy the following conditions

$$xz = 0, \quad zy = 0, \quad x^2 = y^2,$$

and the proof is complete .$\diamond$

Let M be a differential manifold and $T_M = \bigcup_{a \in M} T_aM$, where T_aM is the tangent space of M at a. Let

$$\pi_M : T_M \to M$$

be the canonical projection such that for all $v \in T_aM$ we have $\pi_M(v) = a$. In this case, there is a unique differentiable structure on T_M such that

$\tau_M = (T_M, \pi_M, M, \mathbf{R}^m)$ is a vector bundle ([5], vol.I, chap.3), τ_M is called the *tangent bundle of M.*
In the proof of the theorem 5.3 in [4], we give a strong isomorphism between $\tau_{G/K} = (T_{G/K}, \pi_{G/K}, G/K, \mathbf{R}^m)$ and

$$\xi = (G \times_K \mathcal{G}/\mathcal{K}, \rho_\xi, G/K, \mathcal{G}/\mathcal{K}),$$

then we have

Corollary 6.2. *Let G be the matrix group of all matrices of the form*

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\tau_G = (T_G, \pi_G, G, \mathbf{R}^3)$$

be the tangent bundle of homogeneous Riemannian manifold G. Then a vector W in $\boldsymbol{R}^3$ is a homogeneous vector if and only if its component (x, y, z) satisfy the following conditions

$$xz = 0, \quad zy = 0, \quad x^2 = y^2.$$